University of Rhode Island
DigitalCommons@URI

Nonequilibrium Statistical Physics

Physics Course Materials

2015

# 05. Random Variables: Applications

Gerhard Müller
*University of Rhode Island*, gmuller@uri.edu



Abstract
Part five of course materials for Nonequilibrium Statistical Physics (Physics 626), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.
Updated with version 2 on 5/3/2016.

# Contents of this Document [ntc5]

## [nex14] Reconstructing probability distributions

Determine three probability distributions $P_X(x)$ from the following information:
(a) $\langle X^n \rangle = a^n n!$ for $n \geq 0$,
(b) $\langle\langle X^n \rangle\rangle = a^n (n-1)!$ for $n \geq 1$,
(c) $\langle X^n \rangle = a^n/(n+1)$ for even $n$ and $\langle X^n \rangle = 0$ for odd $n$.

**Solution:**

## [nex95] Probability distribution with no mean value

Consider the function $P_X(x) = x^{-1}e^{-x}I_1(x)$ for $0 < x < \infty$, where $I_1(x)$ is a modified Bessel function.
(a) Show that $P_X(x)$ is normalized to unity.
(b) Produce a plot of $P_X(x)$ for $0 < x < 6$.
(c) Show that a mean value $\langle x \rangle$ does not exist.
(d) Calculate the median value $x_m$ from $\int_0^{x_m} dx P_X(x) = 1/2$.

**Solution:**

## [nex20] Variances and covariances.

A stochastic variable $X$ can have values $x_1 = 1$ and $x_2 = 2$ and a second stochastic variable $Y$ the values $y_1 = 2$ and $y_2 = 3$. Determine the variances $\langle\langle X^2\rangle\rangle$, $\langle\langle Y^2\rangle\rangle$ and the covariance $\langle\langle XY\rangle\rangle$ for two sets of joint probability distributions as defined in [nln7]:
(i) $P(x_1, y_1) = P(x_1, y_2) = P(x_2, y_1) = P(x_2, y_2) = \frac{1}{4}$.
(ii) $P(x_1, y_1) = P(x_2, y_2) = 0, \quad P(x_1, y_2) = P(x_2, y_1) = \frac{1}{2}$.

**Solution:**

## [nex23] Statistically independent or merely uncorrelated?

Consider a classical spin, described by a 3-component unit vector

$$\mathbf{S} = (S_x, S_y, S_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Let us assume that the spin has a completely random orientation, meaning a uniform distribution on the unit sphere. Show that the stochastic variables $\cos\theta, \phi$ are uncorrelated and statistically independent, whereas the stochastic variables $S_x, S_z$ are uncorrelated but not statistically independent. This difference is testimony to the special role of canonical coordinates (here $\cos\theta, \phi$) in statistical mechanics.

**Solution:**

## [nex96] Sum and product of uniform distributions

Consider two independent random variables $X_1, X_2$, both uniformly distributed on the interval $0 < x_1, x_2 < 1$: $P(x_i) = \theta(x_i)\theta(1 - x_i)$, $i = 1, 2$, where $\theta(x)$ is the Heaviside step function. Use transformation relations from [nln49] to calculate range and probability distribution of
(a) the random variable $Y = X_1 + X_2$,
(b) the random variable $Z = X_1 X_2$.
Check the normalization in both cases. Plot $P_Y(y)$ and $P_Z(z)$.

**Solution:**

## [nex79] Exponential integral distribution

Consider two independent random variables $X_1, X_2$, one exponentially distributed, $P_1(x_1) = e^{-x_1},\ 0 < x_1 < \infty$, and the other uniformly distributed, $P_2(x_2) = 1,\ 0 < x_2 < 1$.
(a) Determine the probability distribution $P_Z(z)$ of the random variable $Z = X_1X_2$ for $0 < z < \infty$.
(b) Determine the asymptotic properties of $P_Z(z)$ for $z \to 0$ and for $z \to \infty$.
(c) Calculate the moments $\langle z^n \rangle$ of $P_Z(z)$.
(d) Plot $P_Z(z)$ for $0 < z < 6$.

**Solution:**

## [nex80] Generating exponential and Lorentzian random numbers

Given is a sequence of uniformly distributed random numbers $x_1, x_2, \ldots$ with $0 < x_i < 1$ as produced by a common random number generator.
(a) Find the transformation $Z = Z(X)$ which produces a sequence of random numbers $z_1, z_2, \ldots$ with an exponential distribution:

$$P_Z(z) = \frac{1}{\zeta}\, e^{-z/\zeta}, \quad \zeta > 0.$$

(b) Find the transformation $Y = Y(X)$ which produces a sequence of random numbers $y_1, y_2, \ldots$ with a Lorentzian distribution:

$$P_Y(y) = \frac{1}{\pi}\frac{a}{y^2 + a^2}, \quad a > 0.$$

**Solution:**

## [nex5] Random chords (Bertrand's paradox)

Consider a circle of unit radius and draw *at random* a straight line intersecting it in a chord of length $L$
(a) by taking lines through an arbitrary fixed point on the circle with random orientation;
(b) by taking lines perpendicular to an arbitrary diameter of the circle with the point of intersection chosen randomly on the diameter;
(c) by choosing the midpoint of the chord at random in the area enclosed by the circle.
For each *random choice* determine the probability distribution $P(L)$ for the length of the chord and calculate the average length $\langle L\rangle$.

**Solution:**

## [nex8] From Gaussian to exponential distribution

A random variable $X$ has a continuous Gaussian distribution $P_X(x)$ with mean value $\langle X\rangle = 0$ and variance $\langle\langle X^2\rangle\rangle = 1$. Find the distribution function $P_Y(y)$ for the stochastic variable $Y$ with values $y = x_1^2 + x_2^2$, where $x_1, x_2$ are independent realizations of the random variable $X$. Calculate the mean value $\langle Y\rangle$ and the variance $\langle\langle Y^2\rangle\rangle$.

**Solution:**

## [nex78] Transforming a pair of random variables

Consider two independent random variables $X_1, X_2$ that are uniformly distributed on the intervals $0 \leq x_1, x_2 \leq 1$. Show that the transformed variables

$$Y_1 = \sqrt{-2\ln X_1}\cos 2\pi X_2, \quad Y_2 = \sqrt{-2\ln X_1}\sin 2\pi X_2$$

obey independent normal distributions:

$$P_{\mathbf{Y}}(y_1, y_2) = \frac{1}{\sqrt{2\pi}}\, e^{-y_1^2/2}\, \frac{1}{\sqrt{2\pi}}\, e^{-y_2^2/2}.$$

**Solution:**

## [nex3] Gaussian shootist versus Lorentzian shootist

The shots of two marksmen on a square-shaped target of dimensions 20cm×20cm are found to be distributed with probability densities

$$P_1(x,y) = C_1 e^{-(x^2+y^2)}, \qquad P_2(x,y) = \frac{C_2}{1+x^2+y^2},$$

where $r = \sqrt{x^2+y^2}$ is the distance from the center of the target, and $C_1, C_2$ are normalization constants. Answer the following questions separately for each marksman.
(a) What is the probability that a given shot that hits the target is at least 1cm high ($y > 1$cm)?
(b) Given that a shot that hits the target is at least 1cm high ($y > 1$cm), what is the probability that it is also at least 1cm to the right ($x > 1$cm)?

**Solution:**

## [nex16] Moments and cumulants of the Poisson distribution.

Calculate the generating function $G(z) \equiv \langle z^n \rangle$ and the characteristic function $\Phi(k) \equiv \langle e^{ikn} \rangle$ for the Poisson distribution

$$P(n) = \frac{a^n}{n!}\, e^{-a}, \quad n = 0, 1, 2, \ldots$$

From $\Phi(k)$ calculate the cumulants $\langle\langle n^m \rangle\rangle$. From $G(z)$ calculate the factorial moments $\langle n^m \rangle_f$ and the factorial cumulants $\langle\langle n^m \rangle\rangle_f$.

**Solution:**

## [nex17] Maxwell velocity distribution

In the original derivation of the velocity distribution $f(v_x, v_y, v_z)$ for a classical ideal gas, Maxwell used the following ingredients: (i) The Cartesian velocity components $v_x, v_y, v_z$ (interpreted as stochastic variables) are statistically independent. (ii) The distribution $f(v_x, v_y, v_z)$ is spherical symmetric. (iii) The mean-square velocity follows from the equipartition theorem. Determine $f(v_x, v_y, v_z)$ along these lines.

**Solution:**

## [nex18] Random bus schedules.

Three bus companies $A, B, C$ offer schedules in the form of a probability density $f(t)$ for the intervals between bus arrivals at the bus stop:

$$A: \quad f(t) = \delta(t - T), \qquad B: \quad f(t) = \frac{1}{T}\, e^{-t/T}, \qquad C: \quad f(t) = \frac{4t}{T^2}\, e^{-2t/T}.$$

(i) Find the probability $P_0(t)$ that the interval between bus arrivals is larger than $t$.
(ii) Find the mean time interval $\tau_B$ between bus arrivals and the variance thereof.
(iii) Find the probability $Q_0(t)$ that no arrivals occur in a randomly chosen time interval $t$.
(iv) Find the probability density $g(t)$ of the time a passenger waits for the next bus from the moment he/she arrives at the bus stop.
(v) Find the average waiting time $\tau_P$ of passengers and the variance thereof.

**Solution:**

## [nex106] Life expectancy of the young and the old

The distribution of lifetimes in some population is $f(t) = (4t/T^2)e^{-2t/T}$.
(a) Show that $f(t)$ is properly normalized and that the parameter $T$ is the average lifetime of individuals.
(b) Calculate the conditional probability distribution $P_c(t|\tau)$ for the remaining lifetime of individuals of age $\tau$. Use the expression constructed in [nex38].
(c) If we define the *life expectancy* $T_\tau$ as the average remaining lifetime for an individual of age $\tau$ calculate $T_\tau$ as a function of $T$ and $\tau$.
(d) What is the life-expectancy ratio $T_\infty/T_0$ of the very old and the very young.

**Solution:**

## [nex38] Life expectancy of the ever young

The probability distribution of lifetimes in some population is $f(t)$ with an average lifetime

$$T = \int_0^\infty dt\, t\, f(t)$$

for individuals.
(a) Show that the conditional probability distribution for the remaining lifetime of individuals of age $\tau$ is

$$P_c(t|\tau) = \frac{f(t)}{C(\tau)}\theta(t-\tau), \quad C(\tau) \doteq \int_\tau^\infty dt\, f(t),$$

where $\theta(t)$ is the Heaviside step function.
(b) If we define the *life expectancy* $T_\tau$ as the average remaining lifetime for an individual of age $\tau$ express $T_\tau$ in terms of $P_c(t|\tau)$.
(c) Find the function $f(t)$ for a population (e.g. free neutrons) whose life expectancy is independent of the age of the individual, i.e. for the case where $T_\tau = T$ holds. Then infer an explicit expression for the conditional probability distribution $P_c(t|\tau)$.

**Solution:**

## [nex35] Random frequency oscillator

Consider a physical ensemble of classical harmonic oscillators with randomly distributed angular frequencies: $P_\Omega(\omega) = \frac{1}{2}\Theta(1 - |\omega|)$. At time $t = 0$ all oscillators are excited in phase with unit amplitude: $Y(t) = \cos(\omega t)$.
(a) Find the average displacement $\langle Y(t)\rangle$ and its variance $\langle\langle Y^2(t)\rangle\rangle$ as functions of $t$. What are the long-time asympotic values of these two quantities?
(b) Find the autocorrelation function $\langle Y(t + \tau)Y(t)\rangle$ for arbitrary $t, \tau$ and its asymptotic $\tau$-dependence for $t \to \infty$.
(c) Show that the probability distribution of $Y$ for $m\pi \leq t < (m + 1)\pi$ is

$$P(y, t) = \frac{m}{t\sqrt{1 - y^2}}\,\Theta(1 - |y|) + \frac{1}{t\sqrt{1 - y^2}}\,\Theta(y_{max} - y)\Theta(y - y_{min}),$$

where $y_{max} = 1$, $y_{min} = \cos t$ if $m = 0, 2, 4, \ldots$ and $y_{max} = \cos t$, $y_{min} = -1$ if $m = 1, 3, 5, \ldots$. Find the asymptotic distribution $P(y, \infty)$.

**Solution:**